Mean-field model of free-cooling inelastic mixtures

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We consider a mean-field model describing the free-cooling process of a two-component granular mixture, a generalization of the so called scalar Maxwell model. The cooling is viewed as an ordering process and the scaling behavior is attributed to the presence of an attractive fixed point at $v=0$ for the dynamics. By means of asymptotic analysis of the Boltzmann equation and of numerical simulations we get the following results: (1) we establish the existence of two different partial granular temperatures, one for each component, which violates the zeroth law of thermodynamics; (2) we obtain the scaling form of the two distribution functions; (3) we prove the existence of a continuous spectrum of exponents characterizing the inverse-power-law decay of the tails of the velocity, which generalizes the previously reported value of 4 for the pure model; (4) we find that the exponents depend on the composition, masses, and restitution coefficients of the mixture; (5) we also remark that the reported distributions represent a dynamical realization of those predicted by the nonextensive statistical mechanics, in spite of the fact that ours stem from a purely dynamical approach.

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I. INTRODUCTION

One of the basic principles over which the thermodynamics is built derives from the simple observation that when two systems A and B are brought together so that they can exchange energy (or in thermal contact), after some time they reach a mutual macroscopic equilibrium state in the absence of energy sources, i.e., their temperatures become equal. A second observation is that the thermal equilibrium between a third system C and A implies the thermal equilibrium between B and C. These two facts, which represent the content of the zeroth law of thermodynamics, give sense to the concept of temperature and allow us to build thermometers [1]. The simplest state of aggregation of matter, namely, the gaseous state, offers a neat example of the validity of such a principle. When in equilibrium the state of a mixture of different gases is characterized by a single temperature, i.e., every species has the same average kinetic energy per particle regardless its molecular nature.

An interesting question related to the zeroth law arises quite naturally when we consider the behavior of granular gases, as the assemblies of moving grains at low density are usually called. More in general, granular matter represents a relatively new and unexplored area of research that has attracted the attention of physicists. Just like ordinary matter granular matter may appear under different guises: solidlike or dense aggregates of particles where the motion of its constituents is negligible, liquidlike when a dense aggregate flows, and as a gas when the mutual distances between grains are larger that their typical linear size. Of course, such a classification is empirical but quite loose, since we are not allowed to employ concepts such as thermodynamic phases; in fact granular matter does not fall into the realm of thermodynamics. In other words, only if we are able to show that the basic laws of thermodynamics work in the case of granular materials we can employ its results. There is an emerging consensus that granular materials do not achieve a proper thermodynamic equilibrium. Hence, even if the temperature of a granular assembly can be suitably defined, it does not have the properties required by standard thermodynamics, as we shall discuss below. We shall not digress on some interesting recent proposals aimed to define the temperature of granular packings [2], which is a measure of the potential energy landscape, but we shall confine our attention to dilute systems of granular particles, where the term granular temperature, defined exactly in the same way as in ordinary gases, i.e., as the average of the kinetic energy per particle, has been coined [3]. How far pursue such an analogy? Is the granular temperature a common feature of all granular gases in mutual equilibrium, i.e., the quantity that has the role of determining if two system are in equilibrium with respect to each other with respect to exchanges of energy? If it is not so, the granular temperature has to be regarded just as an ensemble average of the kinetic energy of the grains, which may depend on the microscopic details and, therefore, is spoiled of one of its most useful characteristics.

In the present paper we shall deal with such an issue through a simple model of granular mixture. To be more specific we shall describe a granular mixture by means of a model of the Maxwell type, i.e., a model characterized by a collision rate that is independent of the relative velocity of the colliding particles [4]. Such a class of models has its justification in the case of elastic Maxwell molecules, i.e., particles interacting via a soft repulsive pair potential of the form r^{1-2d}. In the case of inelastic particles the constant collision rate is just a matter of mathematical convenience, since it does not correspond to any known microscopic interaction. In spite of that, it has been demonstrated by several authors [5–9], that the cost of sacrificing physical realism has been widely compensated by the amount of exact analytical results and simplification even in the numerical treatment of the collision process. On the other hand, the real question is, what are the physical consequences of the constant collision rate on our predictions? Some of these are robust with respect to the choice of the model, others will be peculiar, as we shall discuss below. An interesting observation is that the homogeneous cooling states of the Maxwell models and of more realistic models become quite similar if one measures the respective evolutions in terms of the number of collisions occurred up to a given instant, while they are very different in the presence of spatial inhomogeneities [8,10–12]. In the case of granular mixtures there is another aspect that is robust with respect to the choice of the model, namely, the existence of a two temperature behavior. In fact, this has already been observed by Garzó and Dufty [13] in a study of the cooling of a mixture based on the Enskog approximation and experimentally by Feitosa and Menon [14] in a mixture subject to external driving. The Maxwell scalar model, we shall study, also shows such a behavior, but offers the advantage that being considerably simpler than other approaches, it provides a guidance to understand the global aspect of the problem.

Summarizing, the choice of the model is deliberately minimalistic for the following three important reasons: (a) the simplicity of the model allows us to write the exact master equation so that none of the results we find can be ascribed to a breakdown of some particular assumption employed in its derivation; (b) most of the work can be performed analytically and even the form of the asymptotic solutions of the master equation can be estimated in closed manner with a fairly good accuracy; and (c) our solutions can be validated by simple simulations of the microscopic collision process.

Perhaps it is worth mentioning that as an unexpected byproduct of our study we have found a rich variety of behaviors of the velocity distributions as a function of the microscopic control parameters. The common feature of all these distributions is the presence of inverse-power-law high velocity tails.

Incidentally, the observed inverse-power laws for the distributions are identical to those predicted by the nonextensive statistical mechanics (NESM) approach [15,16], that apart from some exceptions still lacks dynamical foundations. An important point to stress is the fact that the observed behavior derives from the solution of the Boltzmann transport equation for a system clearly inspired to the physics of granular materials and already employed in the past, and not the result of assumptions about the form of the generalized entropy, as we shall see below.

The structure of the present paper is the following: in Sec. II we introduce the model, define the notation, and write the Boltzmann equation; in Sec. III we set up the moment expansion and characterize by a granular temperature the macroscopic state of each component; Secs. IV and V, in which we discuss how the scaling solutions for the Boltzmann equation can be obtained, contain all technical details; in Sec. VI we comment the connections between our results and the NESM approach; in Sec. VII we present our conclusions.

II. DEFINITION OF THE MODEL

We shall consider an assembly of N_1 particles of species 1 and N_2 particles of species 2 endowed with scalar velocities v_i^α, with $\alpha=1,2$. The two species may have different masses, m_1 and m_2 and/or different restitution coefficients r_{11}, r_{22}, where $r_{12}=r_{21}$.

The main assumptions of the model are:

(a) the forces are pairwise and impulsive, so that their velocities change at each collision according to the following rule:

$$v_i'^{\alpha}=v_i^\alpha-[1+r_{\alpha\beta}]\frac{m_\beta}{m_\alpha+m_\beta}(v_i^\alpha-v_j^\beta), \tag{1a}$$

$$v'^{\beta}_j=v_j^\beta+[1+r_{\alpha\beta}]\frac{m_\alpha}{m_\alpha+m_\beta}(v_i^\alpha-v_j^\beta), \tag{1b}$$

(b) collisions involving simultaneously more than two particles are disregarded;

(c) within the spirit of mean-field models all pairs are allowed to exchange momentum regardless of their mutual positions.

In Eqs. (1) the primed quantities are the postcollisional velocities and are a linear combination of the velocities before the collisions. The process conserves the number of particles of each species and the total impulse $\Pi=\sum_{\alpha=1,2}\sum_{i=1}^{N_\alpha}m_\alpha v_i^\alpha$.

To study the statistical behavior of the system we consider the coupled equations for the velocity probability distribution functions $P_\alpha(v,t)$, which give the probability density of finding a particle of species α with velocity v at the instant t. In the absence of driving forces we write the following two component Boltzmann equation:

$$\partial_t P_i(v_1,t)=\sum_{j=1}^{2}\int dv_2\left[\frac{1}{\alpha}P_i(v_1'')P_j(v_2'')-P_i(v_1)P_j(v_2)\right]. \tag{2}$$

The system of Boltzmann equations (2) describes the evolution of the pair distribution function (PDF) of the velocities in the case of N_1 and N_2 going to infinity. From a rigorous point of view correlations can appear in a finite system even if such a mean-field approach is employed.

Eliminating the precollisional velocities v_1'',v_2'' in favor of the postcollisional velocities by means of Eq. (1) we obtain the nonlinear system:

$$\begin{aligned}\partial_t P_1(v,t)=&-P_1(v,t)\\&+\frac{2p}{1+r_{11}}\int du\, P_1(u,t)P_1\left(\frac{2v-(1-r_{11})u}{1+r_{11}},t\right)\\&+\frac{(1-p)}{1+r_{12}}\frac{m_1+m_2}{m_2}\int du\, P_1(u,t)P_2\\&\times\left(\frac{\frac{m_1+m_2}{m_2}v-\left(\frac{m_1}{m_2}-r_{12}\right)u}{1+r_{12}},t\right),\end{aligned} \tag{3a}$$

$$\partial_t P_2(v,t) = -P_2(v,t) + \frac{2(1-p)}{1+r_{22}} \int du\, P_2(u,t) P_2\left(\frac{2v-(1-r_{22})u}{1+r_{22}}, t \right) + \frac{p}{1+r_{12}} \frac{m_1+m_2}{m_1} \int du\, P_2(u,t) P_1 \times \left(\frac{\frac{m_1+m_2}{m_1} v - \left(\frac{m_2}{m_1} - r_{12} \right) u}{1+r_{12}}, t \right). \tag{3b}$$

The generalization to many components is straightforward.

To proceed further, it is useful to employ the method of characteristic functions [5,7], i.e., the Fourier transforms of the probability densities defined as

$$\hat{P}_\alpha(k,t) = \int_{-\infty}^{\infty} dv\, e^{ikv} P_\alpha(v,t). \tag{4}$$

The integral equations in Fourier space assume the more convenient form:

$$\partial_t \hat{P}_1(k,t) = -\hat{P}_1(k,t) + p \hat{P}_1(\gamma_{11} k,t) \hat{P}_1((1-\gamma_{11})k,t) + (1-p) \hat{P}_1(\tilde{\gamma}_{12} k,t) \hat{P}_2((1-\tilde{\gamma}_{12})k,t), \tag{5a}$$

$$\partial_t \hat{P}_2(k,t) = -\hat{P}_2(k,t) + (1-p) \hat{P}_2(\gamma_{22} k,t) \hat{P}_2((1-\gamma_{22})k,t) + p \hat{P}_2(\tilde{\gamma}_{21} k,t) \hat{P}_1((1-\tilde{\gamma}_{21})k,t), \tag{5b}$$

where $p = N_1/(N_1+N_2)$, $\zeta = m_1/m_2$, and

$$\gamma_{\alpha\beta} = \frac{1-r_{\alpha\beta}}{2}, \tag{6a}$$

$$\tilde{\gamma}_{12} = \left[1 - \frac{2}{1+\zeta}(1-\gamma_{12}) \right], \tag{6b}$$

$$\tilde{\gamma}_{21} = \left[1 - \frac{2}{1+\zeta^{-1}}(1-\gamma_{12}) \right]. \tag{6c}$$

Interestingly, a mixture of elastic particles of unequal masses does equilibrate under the evolution rule (1), unlike the case $\zeta = 1$. In fact, the solutions of Eq. (5) are the Gaussians $e^{-(\alpha/m_\alpha)k^2}$, and in real space correspond to the Maxwellian velocity distributions.

III. THE MOMENT EXPANSION OF THE DISTRIBUTION FUNCTIONS

For finite inelasticity, let us tentatively assume $\hat{P}_\alpha(k,t)$ to be analytic at the origin $k=0$ and perform a Taylor expansion in powers of k:

$$\hat{P}_\alpha(k,t) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n^{(\alpha)}(t). \tag{7}$$

The coefficient $\mu_n^{(\alpha)}(t)$ of order n of the expansion represents the nth moment of the distribution $P_\alpha(v,t)$ according to the definition in Eq. (4). By substituting the Taylor expansion and equating coefficients, of powers of k, to zero one obtains a set of linear equations for the moments, which can be systematically solved. The moments of order zero are $\mu_0^{(\alpha)}(t) = 1$, due to the normalization of $P_\alpha(v,t)$, while those of first order are solutions of the coupled linear equations:

$$\partial_t \mu_1^{(1)}(t) = -(1-p)(1-\tilde{\gamma}_{12})[\mu_1^{(1)} - \mu_1^{(2)}], \tag{8a}$$

$$\partial_t \mu_1^{(2)}(t) = p(1-\tilde{\gamma}_{21})[\mu_1^{(1)} - \mu_1^{(2)}]. \tag{8b}$$

The total impulse $\Pi = N[p m_1 \mu_1^{(1)} + (1-p) m_2 \mu_1^{(2)}]$ is a constant of motion and corresponds to the eigenvalue $z_1 = 0$ associated with the Galilean invariance, of the linear system (8), whereas the negative eigenvalue $z_2 = -(1 + r_{12})\{[m_1 p + (1-p) m_2]/(m_1+m_2)\}$, describes the exponential law $e^{-z_2 t}$ by which two subsystems 1 and 2 in relative motion with respect to each other have the same average velocity.

The second order cumulants represent the most important statistical indicator of the state of a granular gas and are usually taken as the definition of partial granular temperatures. In order to study their evolution one has to solve the following coupled linear equations for the second moments:

$$\partial_t \mu_2^{(1)}(t) = d_{11}^{(2)} \mu_2^{(1)} + d_{12}^{(2)} \mu_2^{(2)}, \tag{9a}$$

$$\partial_t \mu_2^{(2)}(t) = d_{21}^{(2)} \mu_2^{(1)} + d_{22}^{(2)} \mu_2^{(2)}, \tag{9b}$$

where the coefficients are given by

$$d_{11}^{(n)} = -1 + p[\gamma_{11}^n + (1-\gamma_{11})^n] + (1-p)[\tilde{\gamma}_{12}]^n, \tag{10a}$$

$$d_{12}^{(n)} = (1-p)[(1-\tilde{\gamma}_{12})]^n, \tag{10b}$$

$$d_{22}^{(n)} = -1 + (1-p)[\gamma_{22}^n + (1-\gamma_{22})^n] + p[\tilde{\gamma}_{21}]^n, \tag{10c}$$

$$d_{21}^{(n)} = p[(1-\tilde{\gamma}_{21})]^n. \tag{10d}$$

The solution of the system (9) is a linear combination of real exponentials, which can be expressed as

$$\mu_2^{(\alpha)}(t) = A_\alpha e^{\lambda_1 t} + B_\alpha e^{\lambda_2 t}, \tag{11}$$

where λ_1 and λ_2 are, respectively, the less negative and the more negative eigenvalues of the secular equation associated with the linear system (9). Their computation is rather tedious and due to the presence of many control parameters not particularly illuminating, apart from simple limiting cases, for which one can extract useful information. For this reason in most cases we shall give their values by solving numerically the associated secular equation. An example of the dependence of the larger eigenvalue λ_1 from the control parameters is shown in Fig. 1.

For vanishing first moments the global granular temperature can be defined as

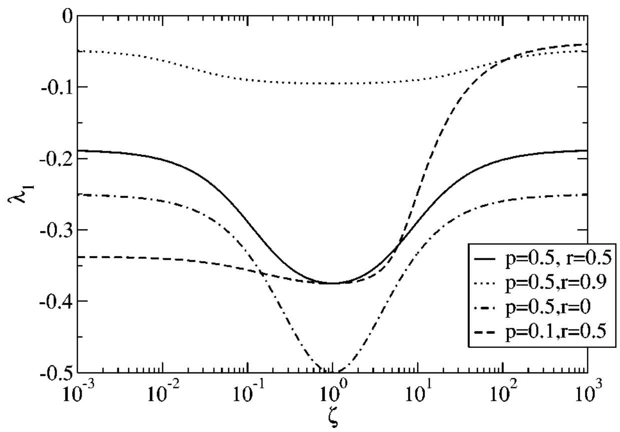

FIG. 1. Maximum eigenvalue λ_1 of the coefficient matrix of Eq. (9) for the evolution of the second moment, as a function of the mass ratio ζ. Different cases are shown for various choices of parameters, p and $r_{11}=r_{22}=r_{12}=r$.

$$T_g = pT_1 + (1-p)T_2, \tag{12}$$

where $T_\alpha = m_\alpha \mu_\alpha^{(2)}/2$ represents the partial granular temperature of species α.

Notice that in the limit of an elastic system (all $\gamma_{\alpha\beta}=0$, but arbitrary ζ and p) the eigenvalue $\lambda_1=0$ reflects the energy conservation. On the other hand, for nonvanishing inelasticity, i.e., values of $\frac{1}{2}\geqslant \gamma_{\alpha\beta}>0$, the energy is dissipated at a finite rate. Consider, for instance, an arbitrary composition p, but $\gamma_{\alpha\beta}=\gamma$ and $\zeta=1$, a situation that describes the approach to the scaling regime of two subsystems 1 and 2 initially prepared at two different initial granular temperatures. One finds that the "energy" eigenvalue is $\lambda_1=2\gamma(\gamma-1)$, while $\lambda_2=-(1-\gamma^2)$. To appreciate the role of λ_2 let us consider the ratio of the granular temperatures:

$$\frac{T_1}{T_2} = \zeta \frac{A_1}{A_2} \frac{1+\dfrac{B_1}{A_1}e^{(\lambda_2-\lambda_1)t}}{1+\dfrac{B_2}{A_2}e^{(\lambda_2-\lambda_1)t}}. \tag{13}$$

Relation (13) means that the resulting "thermalization" time, viz. the time spent to reach the homogeneous cooling regime is proportional to $\tau=(\lambda_2-\lambda_1)^{-1}=(1-\gamma)^{-2}$, i.e., is related to the difference between the two eigenvalues and attains its maximum for the largest inelasticity ($\gamma=1/2$). The asymptotic value of the temperature ratio is given by:

$$\frac{T_1}{T_2} = \zeta \frac{(d_{11}^{(2)}-d_{22}^{(2)}) + \sqrt{(d_{11}^{(2)}-d_{22}^{(2)})^2 + 4d_{12}^{(2)}d_{21}^{(2)}}}{2d_{21}^{(2)}}. \tag{14}$$

It is clear from Eq. (14) that the two granular temperatures of the homogeneous cooling system are in general different (see Fig. 2) as already observed in the Enskog hard sphere kinetic theory of Ref. [13].

In the case of an equimolar mixture ($p=1/2$) with equal inelasticity parameters ($\gamma_{\alpha\beta}=\gamma$), but arbitrary mass ratio the temperature ratio is particularly simple:

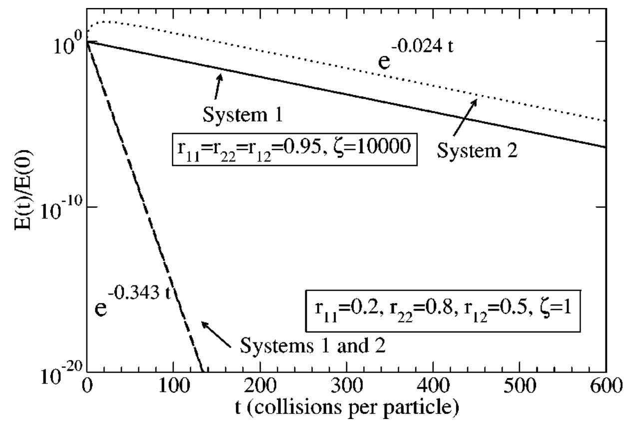

FIG. 2. Plot of the average kinetic energy per particle versus collision time. The two upper curves display the energies of system 1 and system 2, respectively. Notice the initial growth of the average energy of the light species 2. The lowest curve represents a system, whose temperature ratio is 1.

$$\frac{T_1}{T_2} = \frac{(\zeta^2-1)\gamma + \sqrt{(1-\zeta^2)^2\gamma^2 + 4(1-\gamma)^2\zeta^2}}{2(1-\gamma)\zeta}. \tag{15}$$

Notice that the second moment ratio $\rho=\mu_2^{(1)}/\mu_2^{(2)}=T_1/\zeta T_2$ approaches $\gamma/(1-\gamma)$ for $\zeta\to\infty$ and $(1-\gamma)/\gamma$ for $\zeta\to 0$. For small departures from the value $\zeta=1$ a good approximation is:

$$\rho \simeq 1 - \frac{1-2\gamma}{1-\gamma}(\zeta-1), \tag{16}$$

i.e., the temperature ratio increases linearly as a function of ζ and deviates from the equipartition value 1 of an ideal gas mixture for nonvanishing values of the inelasticity parameter γ. In Figs. 3 and 4 we display the moment ratio and the

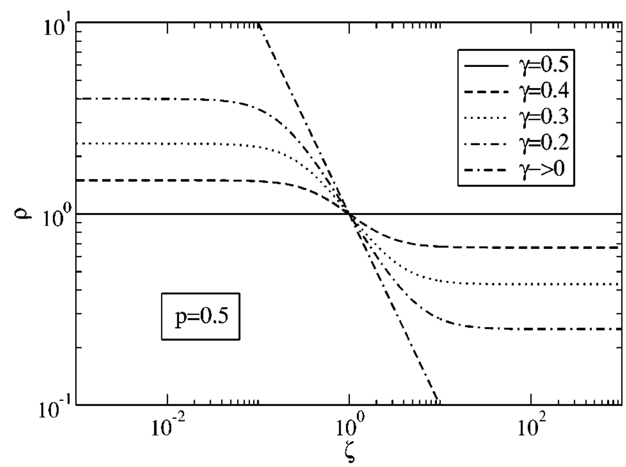

FIG. 3. Asymptotic ratio $\rho=\mu_2^{(1)}/\mu_2^{(2)}$ of second moments, as a function of the mass ratio ζ, for different values of the inelasticity parameter $\gamma_{11}=\gamma_{22}=\gamma_{12}=\gamma=(1-r)/2$, where r is the restitution coefficient. $p=0.5$ for all the curves. The perfectly inelastic case $\gamma=1/2$ corresponds to equal second moments for every value of the mass ratio.

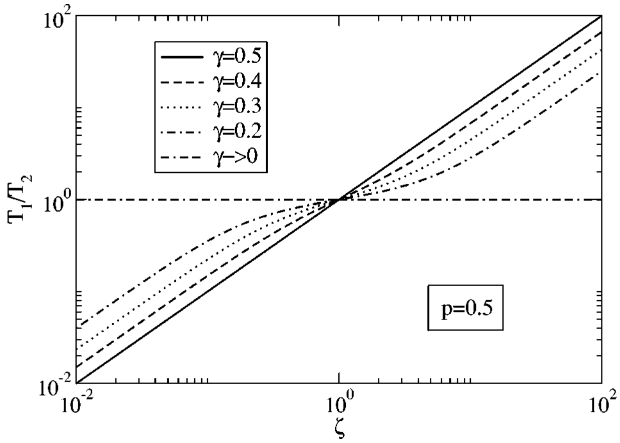

FIG. 4. Ratio $T_1(\infty)/T_2(\infty)$ between asymptotic temperatures of the two components of the mixture, as a function of the mass ratio ζ, for different values of the inelasticity parameter $\gamma_{11}=\gamma_{22}=\gamma_{12}=\gamma=(1-r)/2$, where r is the restitution coefficient. $p=0.5$ for all the curves. The perfectly elastic case is the horizontal line that represents energy equipartition.

temperature ratio, respectively, for different values of the mass ratio and of composition.

When one carries on the evaluation of higher-order moments one discovers the following phenomenon that is the symptom that the moment expansion is ill defined: the rescaled moments beyond that of order m diverge, that is to say the decay of the mth moment is slower than that of the $(m/2)$th power of the second moment. Ben-Naim and Krapivski [7] found that in a pure system, such a phenomenon occurs for all moments beyond the fourth. In other words the kurtosis of the velocity distribution diverges. This is the fingerprint of the presence of an inverse-power-law tail in the velocity distribution function. The situation became clearer after the discovery of a scaling solution [8] as we shall discuss below.

A simple check shows that the state of affairs remains the same in the case of a multicomponent mixture. To show that, we write the evolution equations for the fourth moments:

$$\partial_t \mu_4^{(1)} = d_{11}^{(4)}\mu_4^{(1)} + d_{12}^{(4)}\mu_4^{(2)} + a_{11}(\mu_2^{(1)})^2 + a_{12}\mu_2^{(1)}\mu_2^{(2)}, \tag{17a}$$

$$\partial_t \mu_4^{(2)} = d_{21}^{(4)}\mu_4^{(1)} + d_{22}^{(4)}\mu_4^{(2)} + a_{22}(\mu_2^{(2)})^2 + a_{21}\mu_2^{(1)}\mu_2^{(2)}, \tag{17b}$$

where the coefficients $a_{\alpha\beta}$ are given by

$$a_{11} = 6p[\gamma_{11}(1-\gamma_{11})]^2, \tag{18a}$$

$$a_{22} = 6(1-p)[\gamma_{22}(1-\gamma_{22})]^2, \tag{18b}$$

$$a_{12} = 6(1-p)[(1-\tilde{\gamma}_{12})]^2[\tilde{\gamma}_{12})]^2, \tag{18c}$$

$$a_{21} = 6p[(1-\tilde{\gamma}_{21})]^2[\tilde{\gamma}_{21})]^2. \tag{18d}$$

We have explored the hypersurface $\zeta=1$ of the five dimensional parameter space $\gamma_{\alpha\beta},p,\zeta$ by randomly generating the values of the coupling constants $\gamma_{\alpha\beta}$ and of the composition p and computed the eigenvalues, Λ_1 and Λ_2, with $\Lambda_1 > \Lambda_2$, of the secular equation for the fourth moments, observing that the less negative eigenvalue Λ_1 is larger than twice the eigenvalue λ_1. This is equivalent to say that after a transient the rescaled fourth moment will begin to diverge as in the one-component case. Interestingly, for $\zeta \neq 1$ we observed finite rescaled fourth moments. However, this fact does not imply that for such values of the parameters the moment expansion of the rescaled distribution holds to all orders. It only means that an analogous problem appears at a higher order of the moment expansion, say at the mth moment. We shall elaborate on such an issue in Sec. V.

IV. THE SCALING SOLUTION

A clue to understanding the divergence of the rescaled higher moments came recently from the analysis of the pure case. An exact asymptotic scaling solution of the master equation (3) valid in the asymptotic regime has been found [8]. The merit of such an explicit solution is to show that the characteristic function does not possess a moment expansion, because of the presence of a nonanalyticity at the origin $k=0$ as shown by its representation:

$$f(kv_0(t)) = e^{-v_0|k|}(1+v_0|k|), \tag{19}$$

where $\lim_{t\to\infty}\hat{P}(k,t) = f(kv_0(t))$ and $v_0 = Ae^{-\gamma(\gamma-1)t}$ is the square root of the second moment.

To appreciate such a result one can return to real velocity space where the distribution space is of the form

$$P_s\left(\frac{v}{v_0(t)}\right) = \frac{2}{\pi}\frac{1}{v_0}\frac{1}{\left[1+\left(\frac{v}{v_0}\right)^2\right]^2}, \tag{20}$$

and the moment beyond the third diverge is due to the presence of the inverse-power v^{-4} tail. In fact, it is a general principle that when a singularity of the function in k space approaches the real axis the behavior of the large v components of its Fourier transform turns from exponential to power law [17,18].

An immediate check shows that in the two-component case the above formula fails to provide the correct solution of the master equation, with the exception of some special cases, which correspond to a ratio of the second moments equal to 1.

Let us stress the importance of scaling solutions in the cooling of an inelastic gas. Their existence means that the distribution is fully characterized by its second moment and is self similar, i.e., asymptotically its shape does not change apart from a trivial rescaling. Moreover, it demonstrates that in the cooling problem the distribution functions do not look at all like the distribution functions of an elastic system, which have an infinite number of nondiverging moments. The change induced by the presence of inelasticity is a singular perturbation, viz. an abrupt qualitative modification of the statistical properties of the systems.

Do scaling solutions exist in the case of binary mixtures? To answer such a question let us assume that for a constant cooling rate there exists a scaling solution of the form $\hat{P}_\alpha(k,t)=f_\alpha[\mu_2^{(\alpha)}(t)k^2]$. This implies that the master equation for $\hat{P}$ can be converted into a differential equation with respect to the independent variable k with time independent coefficients. In the one-component case it can be cast in the form:

$$-\alpha k\partial_k\hat{P}(k)+\hat{P}(k)-\hat{P}(\gamma k)P((1-\gamma)k)=0, \qquad (21)$$

with α being a constant. Equation (21) displays the presence of a regular singular point at the origin, $k=0$. Therefore, we look for a particular solution of the form:

$$y(k)=k^\sigma R(k),$$

where σ is the so called indicial exponent and $R(k)$ is a function that is analytic at $k=0$, whose value can be determined by the Frobenius method [19], that provides a systematic tool to compute the solution as a power series of k with noninteger exponents. If $R(k)$ is analytic one can write:

$$y(k)=k^\sigma\sum_{n=0}^{\infty}a_nk^n.$$

However, instead of employing the general method we found it more convenient to extend to the two-component case the abridged version employed by Ernst and Brito to ascertain the nature of the high-velocity tails of the distributions [9,20]. Their technique replaces the small-k Frobenius expansion with its first few terms that contain a contribution nonanalytic at the origin and determining self-consistently the indicial exponent. We assume that the leading singularities of the Fourier transforms are of the form

$$\hat{P}_\alpha(k,t)=1-\frac{1}{2}\mu_2^{(\alpha)}(t)k^2+[\mu_2^{(\alpha)}(t)S_\alpha|k|^2]^{\sigma/2}+[\mu_2^{(\alpha)}(t)]^2Q_\alpha k^4, \qquad (22)$$

with exponent σ and amplitudes Q_α and S_α to be determined by substituting the above approximation into the master equation. The reason for keeping the fourth term will become clear below. Notice that the first two terms in Eq. (22) are fixed, respectively, by the normalization condition and by the variance of the distribution. In fact, up to the second order, Eq. (22) is analytic and identical to the Taylor expansion (7).

Substituting Eq. (22) into Eqs. (3) and equating the like powers of k we obtain the following set of coupled equations:

$$S_1[\partial_t+1-p(\gamma_{11}^\sigma+(1-\gamma_{11})^\sigma)-(1-p)|\tilde{\gamma}_{12}|^\sigma](\mu_2^{(1)})^{\sigma/2}-S_2[(1-p)|1-\tilde{\gamma}_{12}|^\sigma](\mu_2^{(2)})^{\sigma/2}=0, \qquad (23a)$$

$$S_2[\partial_t+1-(1-p)(\gamma_{22}^\sigma+(1-\gamma_{22})^\sigma)-p|\tilde{\gamma}_{21}|^\sigma](\mu_2^{(2)})^{\sigma/2}-S_1[p|1-\tilde{\gamma}_{21}|^\sigma](\mu_2^{(1)})^{\sigma/2}=0. \qquad (23b)$$

To proceed further, we notice that for times much longer than the thermalization time we can safely approximate the second moments with their projection along the eigenvector corresponding to the larger eigenvalue. Doing so the Eqs. (23) reduce to a homogeneous system for the variables S_α. Only in correspondence of special values of the exponent σ the determinant of the coefficients is zero and there exist nonvanishing solutions. In other words we are requiring a solvability condition. The resulting indicial equation is highly nonlinear in the unknown σ, but its numerical solution is straightforward. It reads

$$\left[\frac{\sigma}{2}\lambda_1+1-p[\gamma_{11}^\sigma+(1-\gamma_{11})^\sigma]-(1-p)|\tilde{\gamma}_{12}|^\sigma\right]\times\left[\frac{\sigma}{2}\lambda_1+1-(1-p)[\gamma_{22}^\sigma+(1-\gamma_{22})^\sigma]-p|\tilde{\gamma}_{21}|^\sigma\right]-p(1-p)|1-\tilde{\gamma}_{12}|^\sigma|1-\tilde{\gamma}_{21}|^\sigma=0. \qquad (24)$$

We observe that Eq. (24) has always two solutions $\sigma=0$ and $\sigma=2$ for any choice of the parameters $\gamma_{\alpha,\beta}$, ζ, and p. In fact, these two values correspond to the zeroth and second moment. The nontrivial value $\sigma>2$ represents the indicial exponent associated with the singularity. Somehow surprisingly, we find that such a value of σ is distributed continuously in the interval $2<\sigma<3$ as a function of the control parameters. In other words it means that the inverse-power-law tails of the distribution are sensitive to the composition and to the nature of the interactions in the mixture. We have not found any simple dependence of the exponent σ on the control parameters. Nevertheless, for small asymmetry we observe that σ seems to deviate from the exponent $\sigma=3$ of the pure system quadratically with the temperature ratio.

Knowing just the first three terms of the series (22), apart from the value of the constant $S_1=S_2$, which is still at our disposal, we make the hypothesis that they represent the truncation of the expansion of the following characteristic function:

$$\hat{P}_\alpha(kb_\alpha)=\frac{2}{\Gamma(\nu)}\left(\frac{kb_\alpha}{2}\right)^\nu K_\nu(kb_\alpha), \qquad (25)$$

where $\nu=\sigma/2$ and $K_\nu(z)$ is the modified Bessel function of the second kind of order ν. To render the matter clearer we employ the following Frobenius series representation [21]:

$$\frac{2}{\Gamma(\nu)}\left(\frac{kb_\alpha}{2}\right)^\nu K_\nu(kb_\alpha)=\frac{2}{\Gamma(\nu)}\frac{\pi}{\sin(\pi\nu)}\sum_{n=0}^{\infty}\frac{\left(\frac{kb_\alpha}{2}\right)^{2n}}{n!}\times\left[\frac{1}{\Gamma(n+1-\nu)}-\frac{\left(\frac{kb_\alpha}{2}\right)^{2\nu}}{\Gamma(n+1+\nu)}\right], \qquad (26)$$

considering just the first few terms

$$\hat{P}_\alpha(kb_\alpha)\simeq 1-\frac{1}{(\nu-1)}\left(\frac{kb_\alpha}{2}\right)^2-\left(\frac{kb_\alpha}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}+\left(\frac{kb_\alpha}{2}\right)^4\frac{\Gamma(1-\nu)}{2\Gamma(3-\nu)}. \tag{27}$$

Thus one can see that the first terms of the series (22) and (26) have the same exponents. Therefore, by suitably choosing the coefficients of Eq. (22) we can obtain the Bessel function. In reality, we do so because we are led by the solution of the pure case, which corresponds to $\nu=3/2$. Inserting such a value and employing the following asymptotic expansion (for $z>0$),

$$K_{3/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}(1+z^{-1}),$$

after substitution in Eq. (25) we obtain the solution (19).

If we insist that the identification between the series (22) and the modified Bessel function hold for arbitrary ν we obtain by Fourier transforming Eq. (26) the following approximation to the distribution function [22]:

$$P_\alpha\left(\frac{v}{v_\alpha(t)}\right)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikv}\hat{P}_\alpha(kb_\alpha)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikv}\frac{2}{\Gamma(\nu)}\left(\frac{kb_\alpha}{2}\right)^{\nu}K_\nu(kb_\alpha), \tag{28}$$

i.e.,

$$P_\alpha\left(\frac{v}{v_\alpha(t)}\right)=\frac{1}{b_\alpha\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)}\frac{1}{\left[1+\left(\frac{v}{b_\alpha^2}\right)^2\right]^{\nu+1/2}}, \tag{29}$$

where $v_\alpha^2=b_\alpha^2/[2(\nu-1)]$ represents the second moment.

Such an approximate form clearly displays the inverse-power-law tails with the characteristic exponent $\sigma+1=2\nu+1$. For a pure system since $\sigma\rightarrow 3$ it reduces to the known solution [8]. Numerically we found that such an approximation gives excellent results with an error that is compatible with the uncertainty of the numerical data as shown in Fig. 5. Notice that the indicial equation does not give the value of S_α, but this has been fixed by our ansatz (25). On the other hand we might ask how good is the ansatz. To clarify such an issue we substitute the expansion (22) into the scaling form of the master equation and find two linear inhomogeneous algebraic equations for the two unknown Q_1 and Q_2. Without giving all the details we notice that these equations have certainly nonzero solutions in virtue of the fact observed above, (17) that the eigenvalues of the fourth moment is not twice the eigenvalues of the of the second moment. The numerical solution of the linear system gives two values of the Q's which in general are different ($Q_1\neq Q_2$) for nonspecial values of the parameters, hence the ansatz represents only an approximation. The consequences of this small discrepancy are probably of minor importance for the cases considered in the present section. The deviation of the true series from the series representation of the approximate solution might manifest itself in a small asymmetry in the two rescaled distribution functions, and this might involve the region of small values of v. In principle, there exist the possibility of constructing a better solution via the Frobenius method. However, we believe that one could hardly find a closed form of the distribution functions via an inverse Fourier transform of simplicity comparable to that of Eq. (19). Finally, let us remark that the fairly good agreement between our approximation and the numerically computed solution stems from the fact that the form we propose embodies not only the three following basic ingredients: (i) the normalization condition, (ii) the correct value of the variance, and (iii) the appropriate tails of the distributions, but also some properties of the exact distribution, which one can guess on the basis of pure physical reasoning. These properties are that $P_\alpha(v)$ is symmetric with respect to v, is monotonically decreasing for $v>0$, and is smooth. Such assumptions hugely restrict the class of all possible candidate distributions compatible with the first few terms of the expansion. In practice, we match the small-k behavior with the small r behavior, which is equivalent to the large k behavior.

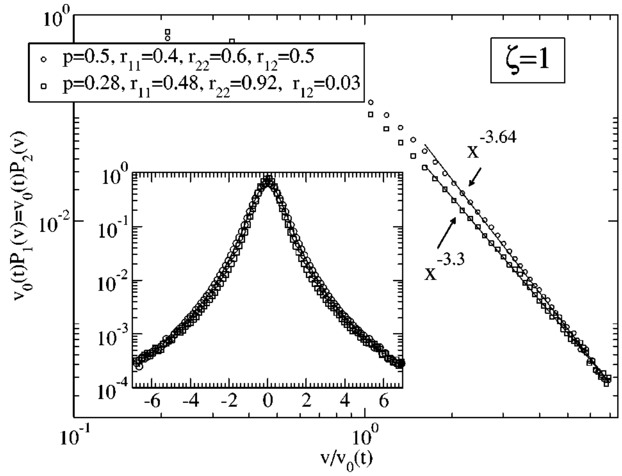

FIG. 5. Rescaled asymptotic velocity distributions from numerical simulations of the model with mass ratio $\zeta=1$ and different values of the restitution coefficients. In the inset the whole distributions are shown, in the main graph the tails are magnified in log-log scale, and fitted with an inverse power law. The predictions obtained by solving the indicial equation are $\sigma=-3.86$ for the upper curve and $\sigma=-3.21$ for the lower curve.

V. HIGHER-ORDER SINGULARITIES

So far we have employed the approximate expansion (22) and found that on the hypersurface $\zeta=1$ the indicial exponent σ lies in the continuous interval $2<\sigma\leqslant 3$. Correspondingly, the distributions possess finite second moments, but diverging fourth moments. On the other hand for $\zeta\neq 1$ as anticipated in Sec. III it is possible to observe different kinds of behaviors. The feature that makes the difference is what

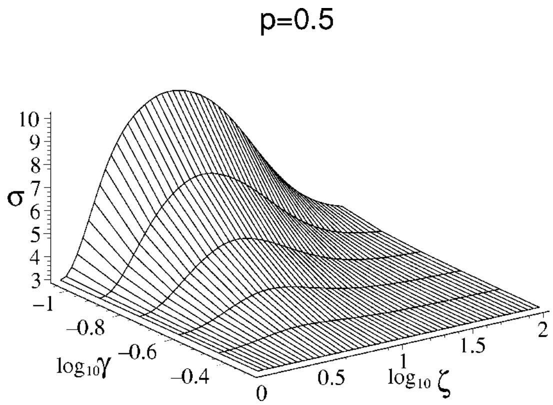

FIG. 6. Power of the singularity σ of the solution of the coupled master equations for the model, as a function of the mass ratio ζ and the inelasticity parameter $\gamma_{11}=\gamma_{22}=\gamma_{12}=\gamma=(-r)/2$, with r the restitution coefficient, with $p=0.5$.

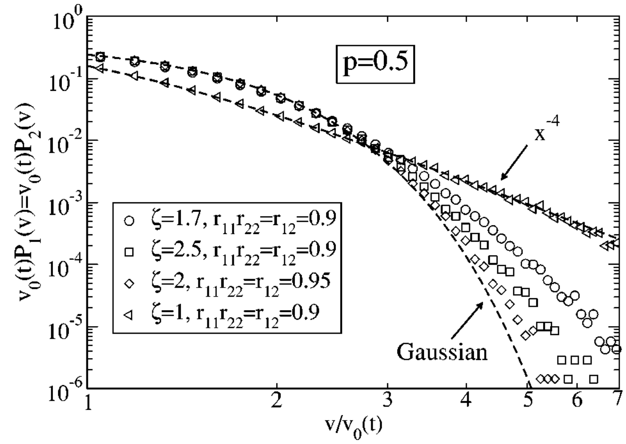

FIG. 7. Tails of the rescaled asymptotic velocity distributions from numerical simulations of the model with number ratio $p=0.5$ and different values of the restitution coefficients and of the mass ratio ζ. The pure case $\zeta=1$ has the exact asymptotic solution $P(x)=(2/\pi)(1+x^2)^{-2}\sim x^{-4}$. The Gaussian is shown as a guide for the eye.

might be called the "isotopic effect," i.e., induced solely by the difference of masses in a binary mixture of particles, otherwise identical. One observes the following phenomenon: all rescaled moments $\mu_s^{(\alpha)}/(\mu_2^{(\alpha)})^{s/2}$ up to the mth order are asymptotically finite, but diverge beyond m.

This is indicative of solutions characterized by very large values of σ. In other words, the allowed interval of values of σ is expanded. This physically means that the exponent σ evolves from a pronounced inverse power law to a Gaussian-like behavior as ζ deviates from unity. Of course, the crossover from one regime to the other is determined by the inelasticity and by the mass ratio in the first place, but the value of the exponent does not depend on any simple way from the control parameters.

Numerically we have considered different coupling parameters and obtained the results shown in Fig. 6 showing the trend that the exponent of the tails increases with decreasing inelasticity and with the difference $|\zeta-1|$.

In Fig. 7 we have plotted the distributions obtained with different values of ζ, showing the change of the exponent of the tails with ζ. We must stress the difficulty of obtaining clear numerical results for the exponents of the power-law tails in the case of large values of the singularity σ (or of the absolute value of the exponent itself, which is $\sigma+1$). In fact, high inverse-power-law tails need far larger statistics to be appreciated; moreover, it appears (numerically) that for large (negative) values of the expected power, the tail appears later in the v/v_0 domain. This means that with a finite statistics one can measure powers smaller, in absolute value, than those expected from the analysis of indicial equations. This phenomena has been observed also in Ref. [9].

To simplify the analysis, let us consider first the master equation for large values of ζ and equal inelasticity parameters. One sees that as $\zeta\rightarrow\infty$ the first equation for the distribution P_1 is asymptotically decoupled from P_2.

$$\lim_{\zeta\rightarrow\infty}\tilde{\gamma}_{12}=1, \tag{30a}$$

$$\lim_{\zeta\rightarrow\infty}\tilde{\gamma}_{21}=2\gamma-1. \tag{30b}$$

For small inelasticity γ, the quantity $\tilde{\gamma}_{21}$ is negative as if the restitution coefficient for the collision (2-1) were larger than one, while $1-\tilde{\gamma}_{12}=0$. Under such conditions, the equation for the distribution function of species 1 becomes effectively decoupled from species 2 as it can be appreciated by substituting Eqs. (30a) and (30b) in Eq. (5):

$$\partial_t\hat{P}_1(k,t)=-\hat{P}_1(k,t)+p\hat{P}_1(\gamma k,t)\hat{P}_1((1-\gamma)k,t) + (1-p)\hat{P}_1(k,t)\hat{P}_2(0,t). \tag{31}$$

Taking into account the fact that $\hat{P}_2(0,t)=1$ one recognizes immediately the equation for the PDF of a pure system, i.e., of the form given by Eq. (20). The decay rate of the energy is given by $\lambda_1=2\gamma(\gamma-1)p$. What happens to the light component? Within this limit the evolution equation for $\hat{P}_2(k,t)$ looks very different from that of $\hat{P}_1(k,t)$. In fact, the numerical solutions indicate the existence of tails with very large values of σ. How can we find this value of the exponent, knowing that the tails of the species 1 diverge as v^{-4}? The trick is to substitute in Eq. (5b) the expansion (22) and neglect the contribution stemming from $\hat{P}_1(k,t)$ because it corresponds to another degree of singularity. At the end one obtains the new indicial equation:

$$2\gamma(\gamma-1)p\frac{\sigma_2}{2}+1-(1-p)[\gamma^{\sigma_2}+(1-\gamma)^{\sigma_2}] + p|1-2\gamma|^{\sigma_2}=0, \tag{32}$$

which for small values of γ has a solution $\sigma_2\simeq 1/[p\gamma(1-\gamma)]$. That is to say for a quasielastic system the exponent of the singularity diverges and all moments exist. To understand such a result we can reconsider the rule (1) and see that

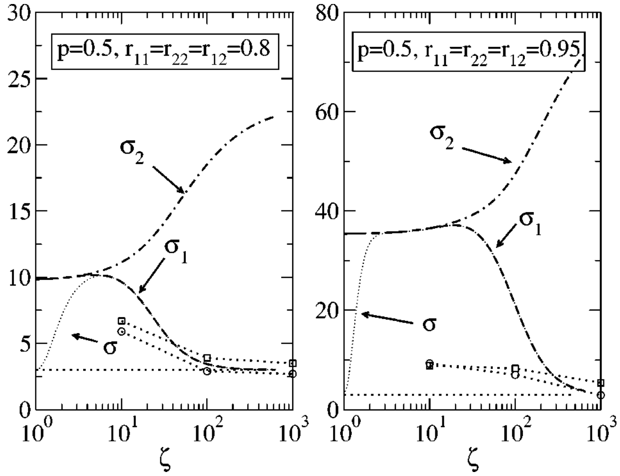

FIG. 8. Indicial exponents σ, σ_1, and σ_2 of the singularity of the solutions of the coupled master equations (5) and of the singularities of the solutions of the noncoupled master equations (31) and (32), respectively, as functions of the mass ratio ζ. The squares and circles refer to the exponents obtained from the numerical simulations (to be compared with σ_1 and σ_2, respectively). The discrepancy with the theoretical predictions is due to the slow convergence, for large σ's, of the tails to the asymptotic value, as discussed in the text.

for $\zeta\to\infty$ the collisions between unlike particles neither change the energy nor the velocity of the heavy species, whereas it changes the velocity of particles 2 in the following way:

$$v'^{(2)}_i = v^{(2)}_i + \zeta(v^{(1)}_j - v^{(2)}_i), \tag{33}$$

i.e., the species 1 acts on 2 as a stochastic noise, since $v^{(1)}$ is randomly distributed according to $P_1(v,t)$. Such a phenomenon is peculiar of the inelastic system under scrutiny and represents a sort of ''reversed Brownian'' motion in which the heavy particles act as a heat bath for the light particles.

In the case of finite ζ we have solved the indicial Eq. (24) and constructed the curves shown in Fig. 8. For small values of the inelasticity γ the surface raises steeply as ζ departs from 1, attains a maximum and then decreases again reaching the asymptotic value $\sigma=3$ for very large values of ζ, as predicted by our asymptotic analysis (see Fig. 6). For larger values of the inelasticity such isotopic effect is less pronounced. This reentrant behavior of σ with ζ is mirrored by an analogous behavior of the rescaled moments (Fig. 9).

To conclude, we remark that for large values of ν the coefficients of the series expansion in powers of v of order $n<\nu$ are very close to the corresponding coefficients of the series expansion of the Gaussian, and only for $n>\nu$ the power-law behavior of the tails becomes manifested. This means that numerically it might be very difficult to detect such a region. Finally for large ν, which correspond to elastic systems, one recovers the Gaussian distribution:

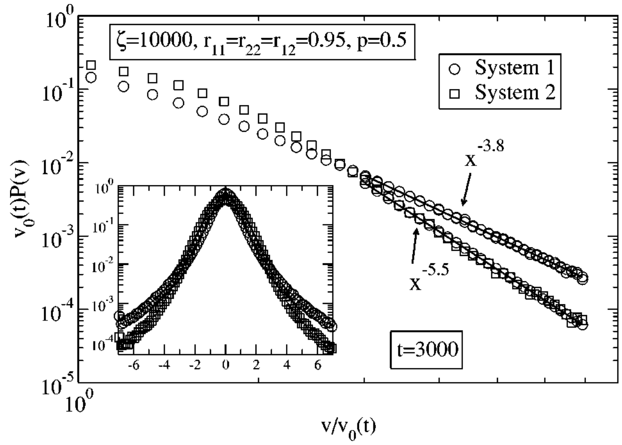

FIG. 9. Plot of the rescaled velocity distributions in the case of a large mass ratio $\zeta=10\,000$. In the inset the two velocity distributions are shown, whereas in the figure the tails are displayed. The tails are characterized by a different inverse power law.

$$\lim_{\nu\to\infty}\frac{1}{b_\alpha\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)}\frac{1}{\left[1+\left(\frac{v}{b_\alpha^2}\right)^2\right]^{\nu+(1/2)}} = \frac{1}{\sqrt{2\pi}}\frac{1}{v_0}e^{-v^2/2v_0^2}. \tag{34}$$

The reentrance phenomenon of the exponent σ deserves a comment. In fact, we have seen that σ controls the tails of the distribution of the heavy component only. In fact, even for ζ to be finite, but larger than some value ζ_{cr}, one observes a bifurcation of the exponents. For $\zeta\geqslant\zeta_{cr}$ the exponent of the heavy component begins to be different from the exponent of the light component. Therefore, our approximation of assuming the same indicial value for both subsystems becomes untenable. Nevertheless, we can find the correct values of the exponents σ_1 and σ_2, by considering that there is no interference between the two singularities associated with k^{σ_1} and k^{σ_2} and, therefore, we get two decoupled indicial equations. The resulting scenario is quite intriguing. After the bifurcation we have two separate trajectories obtained by drawing the values of σ_1 and σ_2 against ζ. Correspondingly, the probability distribution functions of the two subsystems belong to two different critical hypersurfaces. In the limit of $\gamma\to0$ one subsystem flows to the Gaussian elastic fixed point, the other flows to the inelastic fixed point.

VI. RELATION TO NONEXTENSIVE STATISTICAL MECHANICS

We have discussed the cooling behavior of a simple model of granular material and found that the process gives rise to scaling forms of the probability distribution function. Somehow surprisingly, the distribution function has an inverse-power-law decay for large velocities of the form

$v^{-\sigma-1}$ with an exponent σ that depends in a nontrivial fashion on the various control parameters and can vary in the interval $2<\sigma<\infty$ in a continuous way. It is an intriguing fact that such inverse-power-law-like behavior looks undeniably similar to that emerging in the context of the so called nonextensive statistical mechanics [15]. To the best of our knowledge, no other model of comparable simplicity to the present yields similar laws [23] as a result of a dynamical process. We stress again that none of our results has been derived from assumptions about the functional form of a generalized entropy and an associated maximum entropy principle. Our results follow from a completely independent approach, namely, the exact treatment of inelastic collisions based on the master equation.

Classically the temperature, besides being the average of the kinetic energy, is also the intensive thermodynamic field conjugated to the entropy and the latter in turn is related to the distribution function. This is not the case of the granular temperature, which does not possess a definition via the entropic route, but only via the kinetic route. In NESM, instead, one defines a generalized entropy,

$$S_q=\frac{1}{q-1}\left(1-\sum_i p_i^q\right),$$

where the p_i are the probabilities associated with the microstates i of the system. Therefore, S_q does not have a unique expression as in the classical physics, but varies with the exponent q connected to σ by the relation:

$$q=1+\frac{2}{\sigma+1}.$$

Thus S_q would be a function of the various couplings of the particular system under scrutiny.

Summarizing, although the NESM is capable to yield our results, it does not provide the value of the exponent q so that one needs an independent procedure to obtain it and compute the appropriate q entropy. This nonuniversality of the q exponent means that one would need a q entropy for each particular mixture. Finally, a challenging situation for which we do not know the answer within the NESM, is the one inspired by the results of Sec. V, where the two components of the mixture are characterized by different power-law tails; even within the same ''experiment'' one has to resort to two different q entropies.

This state of affairs is just the result of the absence of the zeroth law of thermodynamics for granular systems, i.e., the fact that one needs a different thermometer for each particular granular mixture.

Let us anticipate that in the case of a homogeneously heated granular system our model still predicts a two temperature behavior under rather general conditions, but with the partial temperatures not varying in time [24,25]. On the other hand, the velocity distribution function will not show any power-law tails, but will be very similar to the Gaussians and possess all finite moments. Thus, the power-law behavior is a peculiarity of the cooling process and not a necessity of nonextensivity, at least as far as our model is concerned.

VII. CONCLUSIONS

We have seen that the Maxwell inelastic mixture displays a continuous spectrum of exponents characterizing the inverse-power-law tails of the velocity distribution functions. The previously found solution of the pure system $p=1$ with tail exponent $\sigma+1=4$ can now be regarded as a special case. In fact, a whole spectrum of exponents extending from 3 to ∞ exists. A natural question to ask is why the observed exponent σ deviates from the value of the pure system. The naive guess is that, since in the pure case the solution (19) shows the remarkable feature that its functional form is independent of the cooling rate, i.e., of the microscopic inelasticity parameter γ and that the tails decay invariably as v^{-4}, the same should be true in the case of a mixture of particles characterized by different restitution coefficients. On the contrary, in the mixture case the σ exponent varies with the microscopic parameters. Both these contrasting aspects, namely, a single value of the exponent for the pure model and a continuous spectrum of values for the mixture recall the renormalization group theory of critical phenomena, where critical systems having different Hamiltonians (viz. different γ's in our pure case) in the neighborhood of a fixed point share the same exponents. That is to say that the evolution drives all systems belonging to the same critical (hyper)surface towards a common fixed point. This is why all pure systems exhibit identical asymptotic behaviors in spite of having different inelasticity parameter γ [26]. On the other hand, mixtures correspond to systems whose trajectory lies on a hypersurface whose points are attracted towards a fixed point characterized by $\sigma\neq 3$, because of the presence of other scaling fields.

The model can also be viewed as a quench realized rendering suddenly inelastic an initially elastic system. The subsequent cooling process takes the distribution from its elastic Gaussian fixed point to the zero temperature fixed point. The trajectory along which this process occurs is characterized by peculiar values of the distributions. The scaling behavior is associated with the dynamics of the approach to the fixed point.

Why is the scaling solution selected? Let us make the following observation: if we consider a mixture with a large number of identical components, the mixture formalism allows us to describe independently the evolution of different subsystems. The associated secular equation for M components will have M eigenvalues, $M-1$ of them corresponding to the faster modes. The scaling solution corresponds to the slowest mode and is therefore stable, with respect to fluctuations.

Due to the mean-field nature of the model the fluctuations of a given portion of the system can influence all other elements, i.e., are very long ranged. In a more realistic description of the inelastic interaction the scenario illustrated above will not survive asymptotically, because of the onset of spatial fluctuations, i.e., of correlations that tend to erode the high-velocity tails. These tails, on the other hand, might remain observable during the homogeneous cooling regime but will cease after the formation of spatial gradients in the system [11].

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